ELSEVIER Journal of Geometry and Physics 20 (1996) 250–272

JOURNAL OF GEOMETRY AND PHYSICS

The quantized $SU(2)$ Kepler problem and its symmetry group for negative energies

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Received 13 September 1995; revised 23 November 1995

Abstract

In a previous paper, the $SU(2)$ Kepler problem was defined and shown to admit the symmetry group $SU(4)$ for negative energies. This paper is a continuation of the previous one, giving the quantization of the $SU(2)$ Kepler problem. In the complex vector bundle associated with an $SU(2)$ bundle $\mathbb{R}^8 - \{0\} \to \mathbb{R}^5 - \{0\}$, an extension of the Hopf bundle $S^7 \to S^4$, the quantized $SU(2)$ Kepler problem is defined and analyzed along with its eigenvalues and symmetry. This system, a generalization of the hydrogen atom in five dimensions, describes the motion of a particle with isospin in Yang's monopole field together with the Coulomb potential and a centrifugal potential. It will be shown that the quantized $SU(2)$ Kepler problem of negative energy admits a symmetry group $SU(4) \cong Spin(6)$, which is indeed represented unitarily in the negative energy eigenspaces. The infinitesimal generators of the symmetry are found in an explicit form for all energies, negative, zero, or positive. Those generators coming from the subgroup $Sp(2) \cong Spin(5)$ provide the angular momentum operators and the others are viewed as the Runge–Lenz-like operators. According to whether the energy is negative, zero, or positive, the symmetry Lie algebra formed by these generators is shown to be $so(6)$, $e(5)$, or $so(1,5)$, where $e(5)$ is the Lie algebra of the group of motions in $\mathbb{R}^5$.

Subj. Class.: Quantum mechanics
1991 MSC: 53C80, 81Q05, 81R25
Keywords: Hopf bundle; Complex vector bundle; Kepler problem; Symmetry group

1. Introduction

The Hopf bundles have been recognized to have wide application in physics. For example, the map $S^3 \to S^2$ or its extension $\dot{\mathbb{R}}^4 \to \dot{\mathbb{R}}^3$ with $\dot{\mathbb{R}}^4 := \mathbb{R}^4 - \{0\}$ and $\dot{\mathbb{R}}^3 := \mathbb{R}^3 - \{0\}$ is

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SSDI 0393-0440(95)00059-3

applied in order to "regularize" the Kepler problem [1–3]. In celestial mechanics, the map $\dot{\mathbb{R}}^4 \to \dot{\mathbb{R}}^3$ is called the KS (Kustaanheimo–Stiefel) transformation [1].

The $U(1)$ bundle $\dot{\mathbb{R}}^4 \to \dot{\mathbb{R}}^3$ is used to discuss the MIC (McIntosh and Cisneros)–Kepler problem [4], a generalized Kepler problem, which is defined on $T^*\dot{\mathbb{R}}^3$, the cotangent bundle of $\dot{\mathbb{R}}^3$, in classical case and on L_m, the complex line bundle associated with $\dot{\mathbb{R}}^4 \to \dot{\mathbb{R}}^3$, in quantum case [5,6]. In both cases, this system describes a particle moving in Dirac's monopole field along with the Coulomb potential and a centrifugal potential.

The Hopf bundle $S^7 \to S^4$ and its extension $\dot{\mathbb{R}}^8 \to \dot{\mathbb{R}}^5$ with $\dot{\mathbb{R}}^8 := \mathbb{R}^8 - \{0\}$ and $\dot{\mathbb{R}}^5 := \mathbb{R}^5 - \{0\}$ have also physical application. Yang [7] generalized Dirac's monopole field in $\dot{\mathbb{R}}^3$ to a monopole field in $\dot{\mathbb{R}}^5$, and showed that the generalized monopole or $SU(2)$ monopole field is exactly the BPST (Belavin–Polyakov–Schwartz–Tyupkin) solution [8] to the Yang–Mills equation if restricted on S^4 (see also [9]). In this article, the $SU(2)$ monopole field on $\dot{\mathbb{R}}^5$ is referred to as Yang's monopole field. In classical theory, the bundle $\dot{\mathbb{R}}^8 \to \dot{\mathbb{R}}^5$ was already applied in order to define and study the $SU(2)$ Kepler problem and its symmetry [10].

The aim of this paper is to define and study the quantized $SU(2)$ Kepler problem. To this end, the article starts with a review of the $SU(2)$ bundle $\dot{\mathbb{R}}^8 \to \dot{\mathbb{R}}^5$ and proceeds to its associated vector bundle $\mathcal{E}_l$ ($l = 0, \frac{1}{2}, 1, \ldots$) by using a unitary irreducible representation D^l of $SU(2)$. The quantized conformal Kepler problem is then defined in $\dot{\mathbb{R}}^8$ and will be reduced to a quantum mechanical system in $\mathcal{E}_l$. The reduced system is referred to as the quantized $SU(2)$ Kepler problem, which describes a particle of isospin l moving in Yang's monopole field together with the Coulomb potential and a centrifugal potential. If $l = 0$, Yang's monopole field and the centrifugal potential vanish, and hence the quantized $SU(2)$ Kepler problem becomes the usual quantized Kepler problem (or the hydrogen atom) in five dimensions. The eigenvalues of the quantized $SU(2)$ Kepler problem will be found by using the relation between the quantized conformal Kepler problem and the harmonic oscillator both defined in $\dot{\mathbb{R}}^8$. Kibler [11] and Davtyan et al. [12] gave also the eigenvalues only for $l = 0$ by using a similar method.

Yang's monopole field (or the Yang–Mills field of instanton number one) is known to admit the $Spin(5)$ symmetry [13–15], in association with which the angular momentum operators are defined on the complex vector bundle $\mathcal{E}_l$. Yang [7] gave intuitively those operators in terms of local coordinates. This paper gives global expression to the angular momentum operators.

It was already shown in the previous paper [10] that in classical mechanics the $SU(2)$ Kepler problem admits the symmetry group $SU(4)$ for negative energies. This article shows that the quantized $SU(2)$ Kepler problem admits the symmetry group $SU(4)$ for negative energies as well. Further, the infinitesimal generators of the symmetry group $SU(4)$ will be given and extended further to differential operators for all energies, positive, zero, and negative, which act on cross sections in the vector bundle $\mathcal{E}_l$. Those generators coming from the subgroup $Sp(2) \cong Spin(5)$ are the angular momentum operators, and the others are looked upon as the Runge–Lenz-like operators.

The organization of this paper is outlined as follows: Section 2 is concerned with the $SU(2)$ bundle $\dot{\mathbb{R}}^8 \to \dot{\mathbb{R}}^5$. The canonical connection on this bundle is treated explicitly. The curvature of this connection is Yang's monopole field on $\dot{\mathbb{R}}^5$.

Section 3 deals with the complex vector bundles $\mathcal{E}_l$, $l = 0, \frac{1}{2}, 1, \dots$, which are associated with the $SU(2)$ bundle $\dot{\mathbb{R}}^8 \to \dot{\mathbb{R}}^5$ through unitary irreducible representations D^l of $SU(2)$. Cross sections in $\mathcal{E}_l$ are understood as the states of a particle of isospin l. The linear connection associated with the connection dealt with in Section 2 is reviewed together with its curvature.

In Section 4, the standard metric and Laplacian on $\dot{\mathbb{R}}^8$ are both decomposed into the "vertical" and the "horizontal" parts in accordance with the connection defined in Section 2. The horizontal part of the metric projects to the base space $\dot{\mathbb{R}}^5$ to define a conformally flat metric. The decomposition of the Laplacian will make it feasible to express the Hamiltonian operator for the quantized $SU(2)$ Kepler problem in the next section.

In Section 5, the quantized $SU(2)$ Kepler problem is defined in the Hilbert space of square integrable cross sections in $\mathcal{E}_l$, through the reduction of the quantized conformal Kepler problem in eight dimensions. Further, the reduction procedure shows that the eigenvalues of the quantized $SU(2)$ Kepler problem can be obtained from those of the quantized conformal Kepler problem. Incidentally, the negative energy eigenvalues of the quantized conformal Kepler problem can be derived by using the relation between the conformal Kepler problem and the harmonic oscillator in eight dimensions. Thus the negative energy eigenvalues of the quantized $SU(2)$ Kepler problem are obtained.

Section 6 is concerned with $Sp(2)$ symmetry. The $Sp(2) \cong Spin(5)$ acts on $\dot{\mathbb{R}}^8$ as a group of bundle automorphisms of the $SU(2)$ bundle $\dot{\mathbb{R}}^8 \to \dot{\mathbb{R}}^5$. A basis of the infinitesimal generators of this action is given explicitly, each of which is broken up into the vertical and the horizontal parts, according to the canonical connection on $\dot{\mathbb{R}}^8$. Then these generators give rise to operators in the associated complex vector bundle $\mathcal{E}_l$, which turn out to be the angular momentum operators given locally by Yang [7]. In addition, the second-order Casimir operator of the $Sp(2)$ generators is shown to be related with the standard Laplacian on the unit sphere S^7.

Section 7 is concerned with a symmetry group for the quantized $SU(2)$ Kepler problem of negative energy. For obtaining the symmetry group, the following two facts are crucial. The first one is that a practical method for obtaining an energy eigenspace for the quantized $SU(2)$ Kepler problem is to form a space of D^l-equivariant functions (see Section 3 for definition) out of the eigenspace for the quantized conformal Kepler problem and then to pass to the corresponding eigenspace of cross sections in $\mathcal{E}_l$. The other fact is that $SU(8)$ acts on the eigenspace for the quantized conformal Kepler problem of negative energy. This results from the relation between the quantized conformal Kepler problem and the harmonic oscillator. Therefore, finding a symmetry group for the quantized $SU(2)$ Kepler problem of negative energy amounts to finding a subgroup of $SU(8)$ which leaves invariant the space of D^l-equivariant functions. It will turn out that the subgroup is isomorphic with $SU(4) \cong Spin(6)$, being the symmetry group.

Section 8 provides first the infinitesimal generators of the symmetry group $SU(4)$ for negative energy. These generators are then extended to symmetry operators applicable for all energy, which consists of 10 angular momentum operators and five Runge–Lenz-like operators. Then it will be shown that the symmetry Lie algebra of the quantized $SU(2)$

Kepler problem is $so(6)$, $e(5)$, or $so(1,5)$, according to whether the energy is negative, zero, or positive, where $e(5)$ is the Lie algebra of the group of motions in $\mathbb{R}^5$.

In appendices, the infinitesimal generators of $Sp(2)$ discussed in Section 6 and a basis of the Lie algebra $su(4)$ are listed in the explicit form.

2. The $SU(2)$ bundle $\dot{\mathbb{R}}^8 \to \dot{\mathbb{R}}^5$

We treat the quaternion algebra $\mathbf{H}$ in the 2×2 complex matrix form:

$$\mathbf{H} = \left\{ X = \begin{pmatrix} x_0 + \mathrm{i}x_1 & -x_2 + \mathrm{i}x_3 \\ x_2 + \mathrm{i}x_3 & x_0 - \mathrm{i}x_1 \end{pmatrix};\ x_\nu \in \mathbb{R},\ \nu = 0, 1, 2, 3 \right\}. \tag{2.1}$$

It is decomposed into a vector space direct sum:

$$\mathbf{H} \cong \mathbb{R} \oplus su(2), \tag{2.2}$$

where $\mathbb{R}$ and $su(2)$ stand for the real and the imaginary parts of $\mathbf{H}$, respectively. We choose a basis of $su(2)$ to be

$$E_1 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \tag{2.3}$$

and that of $\mathbb{R}$ to be E_0, the 2×2 identity matrix. Note also that, for $X \in \mathbf{H}$,

$$|x|^2 := \sum_{\nu=0}^{3} x_\nu^2 = \det X, \quad XX^{\mathrm{T}} = |x|^2 E_0, \tag{2.4}$$

where x stands for the 4-vector $x = (x_\nu)$ along with the identification of $\mathbf{H}$ with $\mathbb{R}^4$. We denote by the superscript asterisk the Hermitian conjugate. Then, for $X, Y \in \mathbf{H}$, a symmetric and an anti-symmetric forms are defined from the symmetric and the anti-symmetric parts of XY^* to be, respectively,

$$(X|Y)E_0 = \tfrac{1}{2}(XY^* + YX^*), \tag{2.5}$$

$$\gamma(X, Y) = \tfrac{1}{2}(XY^* - YX^*). \tag{2.6}$$

Eq. (2.5) defines the inner product in $\mathbf{H} \cong \mathbb{R}^4$; one has, indeed, $(X|Y) = \sum_{\nu=0}^{3} x_\nu y_\nu$. Note further that γ takes values in $su(2)$. Now, let $SU(2)$ be identified with the unit quaternions in $\mathbf{H}$. Then $SU(2)$ acts on $\mathbf{H}$. For $g \in SU(2)$ and $X, Y \in \mathbf{H}$, the above-defined forms are subject to the transformation

$$(gX|gY) = (X|Y), \tag{2.7a}$$

$$\gamma(gX, gY) = \mathrm{Ad}_g \gamma(X, Y). \tag{2.7b}$$

With the above setting, we are going to review the $SU(2)$ bundle $SU(2) \to \dot{\mathbb{R}}^8 \to \dot{\mathbb{R}}^5$. Let $SU(2)$ act on $\mathbf{H}^2 \cong \mathbb{R}^8$ diagonally:

$$\phi_g : (X, Y) \to (gX, gY). \tag{2.8}$$

For $(X, Y) \neq (0, 0)$, this action is free and proper, so that $\dot{\mathbb{R}}^8 := \mathbb{R}^8 - \{0\}$ is made into an $SU(2)$ bundle with base manifold $\dot{\mathbb{R}}^8/SU(2) \cong \dot{\mathbb{R}}^5$. The projection π of $\dot{\mathbb{R}}^8$ to the base manifold is realized by

$$\pi : (X, Y) \mapsto (2X^*Y, \det X - \det Y) \in \mathbf{H} \times \mathbb{R} \cong \mathbb{R}^5. \tag{2.9}$$

Set $\pi(X, Y) = (W, w_4) \in \mathbf{H} \times \mathbb{R}$, where the entries in W are taken as in (2.1). Then, written out, Eq. (2.9) gives

$$\begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} y_0 & y_1 & y_2 & y_3 & x_0 & x_1 & x_2 & x_3 \\ y_1 & -y_0 & y_3 & -y_2 & -x_1 & x_0 & -x_3 & x_2 \\ y_2 & -y_3 & -y_0 & y_1 & -x_2 & x_3 & x_0 & -x_1 \\ y_3 & y_2 & -y_1 & -y_0 & -x_3 & -x_2 & x_1 & x_0 \\ x_0 & x_1 & x_2 & x_3 & -y_0 & -y_1 & -y_2 & -y_3 \\ x_3 & x_2 & -x_1 & -x_0 & y_3 & y_2 & -y_1 & -y_0 \\ -x_2 & x_3 & x_0 & -x_1 & -y_2 & y_3 & y_0 & -y_1 \\ -x_1 & x_0 & -x_3 & x_2 & -y_1 & y_0 & -y_3 & y_2 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}. \tag{2.10}$$

This is known also as the Hurwitz transformation [16–18]. Further, we obtain, after calculation,

$$||w||^2 := \sum_{k=0}^{4} w_k^2 = (|x|^2 + |y|^2)^2. \tag{2.11}$$

If we restrict $\dot{\mathbb{R}}^8$ to S^7 with $|x|^2 + |y|^2 = 1$, the $SU(2)$ bundle is contracted to the Hopf bundle $SU(2) \to S^7 \to S^4$ on account of (2.11).

We proceed to the canonical connection defined on the $SU(2)$ bundle $\dot{\mathbb{R}}^8 \to \dot{\mathbb{R}}^5$. Let ξ be a vector in $su(2)$. Then ξ gives rise to a fundamental vector field on $\dot{\mathbf{H}}^2 \cong \dot{\mathbb{R}}^8$; $(\xi X, \xi Y)$ at $(X, Y) \in \dot{\mathbf{H}}^2$. By F_a, $a = 1, 2, 3$, we denote the fundamental vector fields associated with E_a, respectively. At every point $q \in \dot{\mathbb{R}}^8$, the vector fields F_a, $a = 1, 2, 3$, span the vertical subspace of the tangent space $T_q(\dot{\mathbb{R}}^8)$. The horizontal subspace is defined at every point $q \in \dot{\mathbb{R}}^8$ to be the orthogonal complement with respect to the standard metric

$$K_q = \sum_{\nu=0}^{3} (\mathrm{d}x_\nu)^2 + \sum_{\nu=0}^{3} (\mathrm{d}y_\nu)^2. \tag{2.12}$$

Thus a canonical connection is introduced on the bundle $\dot{\mathbb{R}}^8 \to \dot{\mathbb{R}}^5$. The vector fields F_a are mutually orthogonal with respect to the metric K. Fortunately, we can extend $\{F_a\}$ to an orthogonal system $\{F_a, H_k\}$, $a = 1, 2, 3, k = 0, \ldots, 4$, on $\dot{\mathbb{R}}^8$, where H_k span the horizontal subspace of $T_q(\dot{\mathbb{R}}^8)$ at every point q. The components of these vector fields are given in the table below:

$$
\begin{array}{cccccccc|c}
F_1 & F_2 & F_3 & H_0 & H_1 & H_2 & H_3 & H_4 & \\
\hline
-x_1 & -x_2 & x_3 & y_0 & y_1 & y_2 & y_3 & x_0 & \partial/\partial x_0 \\
x_0 & -x_3 & -x_2 & y_1 & -y_0 & -y_3 & y_2 & x_1 & \partial/\partial x_1 \\
x_3 & x_0 & x_1 & y_2 & y_3 & -y_0 & -y_1 & x_2 & \partial/\partial x_2 \\
-x_2 & x_1 & -x_0 & y_3 & -y_2 & y_1 & -y_0 & x_3 & \partial/\partial x_3 \\
-y_1 & -y_2 & y_3 & x_0 & -x_1 & -x_2 & -x_3 & -y_0 & \partial/\partial y_0 \\
y_0 & -y_3 & -y_2 & x_1 & x_0 & x_3 & -x_2 & -y_1 & \partial/\partial y_1 \\
y_3 & y_0 & y_1 & x_2 & -x_3 & x_0 & x_1 & -y_2 & \partial/\partial y_2 \\
-y_2 & y_1 & -y_0 & x_3 & x_2 & -x_1 & x_0 & -y_3 & \partial/\partial y_3
\end{array}
\tag{2.13}
$$

The mutual inner products of F_a and H_k are expressed, at every point $q \in \dot{\mathbb{R}}^8$, as

$$
\begin{aligned}
K_q(F_a, F_b) &= (|x|^2 + |y|^2)\delta_{ab}, \quad a, b = 1, 2, 3, \\
K_q(F_a, H_k) &= 0, \\
K_q(H_j, H_k) &= (|x|^2 + |y|^2)\delta_{jk}, \quad j, k = 0, \ldots, 4.
\end{aligned}
\tag{2.14}
$$

Further, through the projection π the vector fields F_a and H_k project to

$$\pi_* F_a = 0, \quad \pi_* H_k = 2\|w\| \frac{\partial}{\partial w_k}, \tag{2.15}$$

respectively. Thus the horizontal lift $(\partial/\partial w_k)^*$ of $\partial/\partial w_k$, the horizontal vector field that projects to $\partial/\partial w_k$, are expressed as

$$\left(\frac{\partial}{\partial w_k}\right)^* = \frac{1}{2(|x|^2 + |y|^2)} H_k, \quad k = 0, \ldots, 4. \tag{2.16}$$

The system $\{F_a, (\partial/\partial w_k)^*\}$ also forms an orthogonal system on $\dot{\mathbb{R}}^8$, which we prefer to use below.

We now turn to the connection form for the canonical connection stated above, which is defined to be

$$\omega = \frac{\gamma(\mathrm{d}X, X) + \gamma(\mathrm{d}Y, Y)}{|x|^2 + |y|^2}, \tag{2.17}$$

where γ is the anti-symmetric form given by (2.6). It is a matter of calculation to show that, for a fundamental vector field $(\xi X, \xi Y)$ and for the $SU(2)$ action,

$$\omega((\xi X, \xi Y)) = \xi, \tag{2.18}$$

$$\phi_g^* \omega = \mathrm{Ad}_g \omega. \tag{2.19}$$

This verifies that ω is indeed a connection form [19]. It is to be noted here that for the radial vector (X, Y), ω vanishes by the definition (2.17), so that ω is contractible to a form on S^7.

Let

$$\omega = \sum_{a=1}^{3} \omega^a E_a, \tag{2.20}$$

where E_a is the basis (2.3) of $su(2)$. Then the system of one-forms $\{\omega^a, \pi^* \mathrm{d}w_k\}$ turns out to be dual to the system of vector fields $\{F_a, (\partial/\partial w_k)^*\}$.

The curvature form Ω for the connection ω is defined as

$$\Omega = \mathrm{d}\omega - \omega \wedge \omega. \tag{2.21}$$

We notice here that the minus sign in the definition is due to the left action of the structure group. The transformation property of Ω is expressed, on account of (2.19), as

$$\phi_g^* \Omega = \mathrm{Ad}_g \Omega. \tag{2.22}$$

With respect to a basis, $(\partial/\partial w_k)^*$, $k = 0, 1, \dots, 4$, of horizontal vector fields, the components of Ω take the form

$$\Omega\left(\left(\frac{\partial}{\partial w_j}\right)^*, \left(\frac{\partial}{\partial w_k}\right)^*\right) = -\omega\left(\left[\left(\frac{\partial}{\partial w_j}\right)^*, \left(\frac{\partial}{\partial w_k}\right)^*\right]\right). \tag{2.23}$$

We notice here that Ω is known as Yang's monopole field on $\dot{\mathbb{R}}^5$ [7,9].

3. The associated complex vector bundles

Let D^l be a unitary irreducible representation of $SU(2)$ with $l = 0, \frac{1}{2}, 1, \dots$, and V_l a representation space for D^l. For $g \in SU(2)$, we denote by $D^l(g)$ the unitary operator on V_l. A left action of $SU(2)$ on the product space $\dot{\mathbb{R}}^8 \times V_l$ is defined by

$$(q, z) \mapsto (\phi_g(q), D^l(g)z), \quad (q, z) \in \dot{\mathbb{R}}^8 \times V_l. \tag{3.1}$$

This action defines an equivalence relation in $\dot{\mathbb{R}}^8 \times V_l$. By $\dot{\mathbb{R}}^8 \times_l V_l$ we mean the quotient space by this equivalence relation. Then the complex vector bundle $\mathcal{E}_l = (\dot{\mathbb{R}}^8 \times_l V_l, \pi_l, \dot{\mathbb{R}}^5)$ is defined so that the following diagram may be commutative:

$$\begin{array}{ccc} \dot{\mathbb{R}}^8 \times V_l & \xrightarrow{q_l} & \dot{\mathbb{R}}^8 \times_l V_l \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi_l} \\ \dot{\mathbb{R}}^8 & \xrightarrow[\pi]{} & \dot{\mathbb{R}}^5 \end{array} \tag{3.2}$$

where q_l is the natural projection and p_1 is the projection to the first factor space.

A map σ of $\dot{\mathbb{R}}^5$ to $\dot{\mathbb{R}}^8 \times_l V_l$ is called a cross section in $\mathcal{E}_l$ if it satisfies $\pi_l \circ \sigma = id$, id being the identity map of $\dot{\mathbb{R}}^5$. Cross sections in $\mathcal{E}_l$ are viewed as states of a particle of isospin l, since each fibre of $\mathcal{E}_l$ is isomorphic with $V_l \cong \mathbb{C}^{2l+1}$. The cross sections are known to be in one-to-one correspondence with the D^l-equivariant functions f on $\dot{\mathbb{R}}^8$ [19], which are defined as V_l-valued functions on $\dot{\mathbb{R}}^8$ satisfying, for $g \in SU(2)$ and $q \in \dot{\mathbb{R}}^8 \cong \dot{\mathbf{H}}^2$,

$$f(\phi_g(q)) = D^l(g) f(q). \tag{3.3}$$

We denote this correspondence by $q_l^\#$. Let $f = q_l^\# \sigma$. Then the cross section σ is expressed as $\sigma(\pi(q)) = [(q, f(q))]$, where the square brackets denote the equivalence class.

The vector bundle $\mathcal{E}_l$ is endowed with the linear connection associated with the connection defined in Section 2. Let ξ be a vector field on $\dot{\mathbb{R}}^5$ and ξ^* the horizontal lift of ξ. Then the linear connection ∇ in $\mathcal{E}_l$ is defined for cross sections σ through

$$\nabla_\xi \sigma = q_l^{\#-1} \xi^* q_l^\# \sigma. \tag{3.4}$$

$\nabla_\xi \sigma$ is referred to as the covariant derivative of σ with respect to ξ. We notice here that Yang's monopole field is already minimally coupled, through ∇_ξ, with the vector field ξ as a first-order differential operator.

The curvature of the connection ∇ is defined for vector fields ξ and η on $\dot{\mathbb{R}}^5$ and a cross section σ by

$$R(\xi, \eta)\sigma = ([\nabla_\xi, \nabla_\eta] - \nabla_{[\xi,\eta]})\sigma. \tag{3.5}$$

Combined with (3.4), the definition (3.5) is expressed as

$$R(\xi, \eta)\sigma = q_l^{\#-1}([\xi^*, \eta^*] - [\xi, \eta]^*)q_l^\# \sigma. \tag{3.6}$$

4. Decomposition of the metric and the Laplacian

In the preceding sections, we have reviewed the $SU(2)$ bundle $\dot{\mathbb{R}}^8 \to \dot{\mathbb{R}}^5$ and its associated vector bundles $\mathcal{E}_l$. We wish to define a quantized $SU(2)$ Kepler problem on $\mathcal{E}_l$. To this end, we are to break up in advance the standard metric and the Laplacian on $\dot{\mathbb{R}}^8$ into horizontal and vertical parts in accordance with the connection defined in Section 2. The breaking-up makes it feasible to express the Hamiltonian operator of the quantized $SU(2)$ Kepler problem in an explicit form.

From (2.14), (2.16), the standard metric K is put in the form

$$K = r \sum_{a=1}^{3} (\omega^a)^2 + \frac{1}{4r} \sum_{k=0}^{4} (\pi^* \mathrm{d}w_k)^2, \quad r = \sqrt{\sum_{k=0}^{4} w_k^2}. \tag{4.1}$$

The metric K provides the inner product in the tangent space at each point q of $\dot{\mathbf{H}}^2 \cong \dot{\mathbb{R}}^8$. In the cotangent space $T_q^*(\dot{\mathbb{R}}^8)$, the dual inner product is defined, which we denote by K_q^*. Then, we obtain, in the form dual to (2.14),

$$\begin{aligned} &K^*(\omega^a, \omega^b) = \frac{1}{r}\delta_{ab}, \\ &K^*(\omega^a, \pi^* \mathrm{d}w_k) = 0, \\ &K^*(\pi^* \mathrm{d}w_k, \pi^* \mathrm{d}w_j) = 4r\delta_{kj}. \end{aligned} \tag{4.2}$$

Now let f be a function on $\dot{\mathbb{R}}^8$. Then the differential $\mathrm{d}f$ turns out to be expressed as

$$\mathrm{d}f = \sum_{a=1}^{3} (F_a f)\omega^a + \sum_{k=0}^{4} \left(\left(\frac{\partial}{\partial w_k} \right)^* f \right) \pi^* \mathrm{d}w_k. \tag{4.3}$$

From (4.2) and (4.3), it follows that

$$K^*(\mathrm{d}f,\,\mathrm{d}f) = \frac{1}{r}\sum_{a=1}^{3}(F_a f)^2 + 4r\sum_{k=0}^{4}\left(\left(\frac{\partial}{\partial w_k}\right)^* f\right)^2, \tag{4.4}$$

which is dual to (4.1).

The metric K projects through π to a metric A on the base space $\dot{\mathbb{R}}^5$; for tangent vectors ξ_1 and ξ_2 to $\dot{\mathbb{R}}^5$ at $\pi(q)$, one has

$$A_{\pi(q)}(\xi_1, \xi_2) := K_q(\xi_1^*, \xi_2^*), \tag{4.5}$$

where ξ_1^* and ξ_2^* are the horizontal lifts of ξ_1 and ξ_2, respectively. This definition is, of course, independent of the choice of q in the fibre $\pi^{-1}(\pi(q))$. Eq. (4.1) then provides

$$A\left(\frac{\partial}{\partial w_j}, \frac{\partial}{\partial w_k}\right) = \frac{1}{4r}\delta_{jk}, \tag{4.6}$$

which shows that A is a conformally flat metric on $\dot{\mathbb{R}}^5$. Further, we denote by B the induced metric on the unit sphere S^4; $B = A|_{S^4}$. Then from (4.6), B is $\frac{1}{4}$ times the canonical metric on S^4. Further, Eq. (4.1) is expressed also as

$$K = r\sum_{a=1}^{3}(\omega^a)^2 + \frac{1}{4r}\,\mathrm{d}r^2 + r\pi^* B. \tag{4.7}$$

From this it follows that the standard volume element $\mathrm{d}x\,\mathrm{d}y$ on $\dot{\mathbb{R}}^8$ with $\mathrm{d}x = \mathrm{d}x_0\cdots\mathrm{d}x_3$ and $\mathrm{d}y = \mathrm{d}y_0\cdots\mathrm{d}y_3$ is put in the form

$$\mathrm{d}x\,\mathrm{d}y = \tfrac{1}{2}r^3\,\mathrm{d}r\,\mathrm{d}S\,\mathrm{d}G, \tag{4.8}$$

where $\mathrm{d}S$ is the volume element on S^4 determined by the metric B, and $\mathrm{d}G$ is the volume element on $SU(2)$, which is given by $\mathrm{d}G = \theta^1\wedge\theta^2\wedge\theta^3$ together with $\mathrm{d}gg^{-1} = \sum\theta^a E_a$.

We turn to the standard Laplacian Δ on $\mathbb{R}^8$. Then, for a function f of compact support on $\mathbb{R}^8$, one has

$$\int_{\mathbb{R}^8} K^*(\mathrm{d}f,\,\mathrm{d}f)\,\mathrm{d}x\,\mathrm{d}y = -\int_{\mathbb{R}^8} f\Delta f\,\mathrm{d}x\,\mathrm{d}y. \tag{4.9}$$

If we substitute the expression (4.4) for $K^*(\mathrm{d}f,\,\mathrm{d}f)$ in (4.9), and carry out integration by part, we obtain the Laplacian in the form

$$\Delta = \frac{1}{r}\sum_{a=1}^{3}(F_a)^2 + 4r\sum_{k=0}^{4}\left(\left(\frac{\partial}{\partial w_k}\right)^*\right)^2, \tag{4.10}$$

which gives the decomposition of the Laplacian into vertical and horizontal parts.

5. The quantized $SU(2)$ Kepler problem

To define the quantized $SU(2)$ Kepler problem, let us be reminded of the quantized MIC–Kepler problem [6], which is defined in the complex line bundle associated with the $U(1)$ bundle $\dot{\mathbb{R}}^4 \to \dot{\mathbb{R}}^3$, through the reduction of the quantized conformal Kepler problem on $\dot{\mathbb{R}}^4$. Since the structure group for the complex line bundle is $U(1)$, we could have called the quantized MIC–Kepler problem the quantized $U(1)$ Kepler problem. With this in mind, we wish to define, in an analogous manner, the quantized $SU(2)$ Kepler problem in the complex vector bundle $\mathcal{E}_l$ associated with the $SU(2)$ bundle $\dot{\mathbb{R}}^8 \to \dot{\mathbb{R}}^5$, by carrying out the reduction procedure for the quantized conformal Kepler problem on $\dot{\mathbb{R}}^8$.

5.1. *The quantized conformal Kepler problem*

The quantized conformal Kepler problem has the Hamiltonian operator [20] defined by

$$\widehat{H} = -\frac{1}{8r}\Delta - \frac{\kappa}{r}, \quad \kappa > 0; \text{const.}, \tag{5.1}$$

where Δ is the standard Laplacian on $\mathbb{R}^8$, and $r = |x|^2 + |y|^2$. The quantized conformal Kepler problem should be defined on the Hilbert space of square integrable complex-valued functions on $\mathbb{R}^8$ with the inner product [20]

$$\langle f, h\rangle = \int_{\mathbb{R}^8} \overline{f} h 4r \,\mathrm{d}x\,\mathrm{d}y. \tag{5.2}$$

Here and in the below, the overbar indicates the complex conjugate. It is an easy matter to show that the Hamiltonian (5.1) is a symmetric operator in $C_0^\infty(\mathbb{R}^8)$ with respect to the inner product (5.2). Quite recently, Trunk [21] introduced the Hurwitz–Kepler problem, the Hamiltonian of which takes the form of $\widehat{H}$ plus some operators concerning $SU(2)$.

The operator (5.1) is linked with the harmonic oscillator Hamiltonian operator

$$\widehat{K}_\lambda = -\tfrac{1}{2}\Delta + \tfrac{1}{2}\lambda^2 r \tag{5.3}$$

with λ a real constant, through the relation

$$4r(\widehat{H} + \tfrac{1}{8}\lambda^2) = \widehat{K}_\lambda - 4\kappa. \tag{5.4}$$

By using this relation, negative energy eigenvalues for $\widehat{H}$ are obtained as follows: For $\widehat{H}$ given, consider $\widehat{K}_\lambda$ with a parameter λ. Let us assume that an eigenvalue of $\widehat{K}_\lambda$ equals 4κ. Then $\widehat{H}$ must have an eigenvalue $-\frac{1}{8}\lambda^2$. Since the eigenvalues of $\widehat{K}_\lambda$ are given by $\lambda(n+4)$, $n = 0, 1, 2, \dots$, one has the relation

$$\lambda(n+4) = 4\kappa, \tag{5.5}$$

so that $-\frac{1}{8}\lambda^2$ turns out to be

$$E_n := -\frac{2\kappa^2}{(n+4)^2}, \quad n = 0, 1, 2, \dots \tag{5.6}$$

Further, Eq. (5.4) implies the following result.

Lemma 5.1. *The eigenfunctions of* $\widehat{K}_\lambda$ *with* $\lambda = 4\kappa/(n+4)$ *are those of* $\widehat{H}$ *associated with the eigenvalue* E_n *given by* (5.6).

We denote by $S(E_n)$ the eigenspace for the quantized conformal Kepler problem of negative energy E_n.

5.2. The quantized $SU(2)$ *Kepler problem*

The quantized conformal Kepler problem can be reduced by letting the Hamiltonian operator $\widehat{H}$ act on the space of D^l-equivariant functions. Let V_l be a carrier space for D^l. We are to endow the space of cross sections in $\mathcal{E}_l$ with an inner product structure. Let σ_1 and σ_2 be cross sections corresponding to D^l-equivariant functions f_1 and f_2, respectively, and let $(\ |\ )$ denote a Hermitian inner product on V_l with respect to which $D^l(g)$ are unitary for all g. Then the function $(f_1|f_2) = (q_l^{\#}\sigma_1|q_l^{\#}\sigma_2)$ on $\dot{\mathbb{R}}^8$ is invariant under the $SU(2)$ action, so that it can be viewed as a function on $\dot{\mathbb{R}}^5$. On the other hand, the standard volume element of $\dot{\mathbb{R}}^5$, denoted by $\mathrm{d}w$, is expressed as $\mathrm{d}w = 16r^4\,\mathrm{d}r\,\mathrm{d}S$, where $\mathrm{d}S$ is the volume element on S^4 determined by the metric B. Then, from (4.8) one has $4r\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{8}\,\mathrm{d}G\,\mathrm{d}w$, so that the inner product of σ_1 and σ_2 is defined as

$$\langle\sigma_2|\sigma_2\rangle := \frac{1}{8}\int_{SU(2)} \mathrm{d}G \int_{\dot{\mathbb{R}}^5} (q_l^{\#}\sigma_1|\,q_l^{\#}\sigma_2)\,\mathrm{d}w = \frac{\pi^2}{4}\int_{\dot{\mathbb{R}}^5} (q_l^{\#}\sigma_1|q_l^{\#}\sigma_2)\,\mathrm{d}w, \tag{5.7}$$

where we have used the fact that the volume of $SU(2)$ is $2\pi^2$ with respect to $\mathrm{d}G$. We denote by Γ_l the Hilbert space of square integrable cross sections with respect to the inner product (5.7).

We now proceed to reduce the Hamiltonian operator (5.1) in order to obtain a Hamiltonian operator acting on Γ_l. Making use of the decomposition (4.10) together with

$$\widehat{F}_a = F_a/2\mathrm{i}, \quad \mathrm{i} = \sqrt{-1}, \tag{5.8}$$

we put $\widehat{H}$ in the form

$$\widehat{H} = -\frac{1}{2}\sum_{k=0}^{4}\left(\left(\frac{\partial}{\partial w_k}\right)^*\right)^2 + \frac{1}{2r^2}\sum_{a=1}^{3}(\widehat{F}_a)^2 - \frac{\kappa}{r}. \tag{5.9}$$

Let f be a D^l-equivariant function. Then by differentiating at $t = 0$ both sides of (3.3) with $g = \exp(\frac{1}{2}tE_a)$, we obtain

$$\widehat{F}_a f = [\widehat{F}_a]f, \tag{5.10}$$

where $[\widehat{F}_a]$ denotes the representation matrix of $\widehat{F}_a$. Since $\sum_{a=1}^{3}[\widehat{F}_a]^2$ equals $l(l+1)$ times the identity matrix, one has

$$\sum_{a=1}^{3}(\widehat{F}_a)^2 f = l(l+1)f. \tag{5.11}$$

From (3.4), (5.9), and (5.11), the reduced Hamiltonian operator $\widehat{H}_l$ defined by

$$\widehat{H}_l = q_l^{\#-1} \widehat{H} q_l^{\#} \tag{5.12}$$

turns out to be expressed as

$$\widehat{H}_l = -\frac{1}{2}\sum_{k=0}^{4} \nabla_k^2 + \frac{l(l+1)}{2r^2} - \frac{\kappa}{r}, \tag{5.13}$$

where $\nabla_k = \nabla_{\partial/\partial w_k}$, $k = 0, \dots, 4$. We call the quantum system defined in $\mathcal{E}_l$ with the Hamiltonian operator (5.13) the quantized $SU(2)$ Kepler problem. The operator $\widehat{H}_l$ is regarded as the quantization of the classical Hamiltonian for the $SU(2)$ Kepler problem [10]. We note again that Yang's monopole field is minimally coupled in the covariant differential operators ∇_k. If $l = 0$, $\widehat{H}_l$ becomes the usual Hamiltonian operator for the quantized Kepler problem (or the hydrogen atom) in $\mathbb{R}^5$.

Theorem 5.2. *By an $SU(2)$ action, the quantized conformal Kepler problem is reduced to the quantized $SU(2)$ Kepler problem defined in the Hilbert space Γ_l of square integrable cross sections in $\mathcal{E}_l$ together with the Hamiltonian $\widehat{H}_l$ given by* (5.13), *where l is a non-negative half integer.*

From the definition (5.12), it follows that the negative eigenvalues of $\widehat{H}_l$ come from those of $\widehat{H}$. In fact, if σ is an eigen-cross section for $\widehat{H}_l$, $q_l^{\#}\sigma$ must be a V_l-valued D^l-equivariant eigenfunction for $\widehat{H}$. A question as to how one finds the space of D^l-equivariant eigenfunctions out of the eigenspace $S(E_n)$ will be investigated in Section 7. Further, we have to be careful in the parity of l. As will be shown also in Section 7, n's are even or odd, according as $l = 0, 1, 2, \dots$, or $l = \frac{1}{2}, \frac{3}{2}, \dots$, together with $n \geq 2l$. Thus we have the following theorem.

Theorem 5.3. *Let $S(E_n; l)$ be the space of D^l-equivariant eigenfunctions, which is a subspace of the eigenspace for the quantized conformal Kepler problem of eigenvalue $E_n = -2\kappa^2/(n+4)^2$. If n is taken to be even or odd according as l is an integer or a half-integer together with $n \geq 2l$, then $S(E_n; l)$ is in one-to-one correspondence with the eigenspace $q_l^{\#-1} S(E_n; l)$ of negative energy E_n for the quantized $SU(2)$ Kepler problem.*

In conclusion, we notice that the eigenvalues E_n are obtained by solving the eigenvalue problem as well. Let h be an eigenfunction of the spherical Laplacian Δ_7 on S^7; $\Delta_7 h = -p(p+6)h$, $p = 0, 1, 2, \dots$ We assume that a function $f = \phi(r)h$ is an eigenfunction which solves the equation $\widehat{H} f = Ef$. Then we obtain the equation for $\phi(r)$,

$$-\frac{1}{2r^4}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^4 \frac{\mathrm{d}\phi}{\mathrm{d}r}\right) + \left(\frac{p(p+6)}{8r^2} - \frac{\kappa}{r}\right)\phi = E\phi. \tag{5.14}$$

If p is an even number, *i.e.*, $p = 2j$, $j = 0, 1, 2, \dots$, this equation can be viewed as that for the hydrogen atom in five dimensions. The negative eigenvalues resulting from (5.14) with $p = 2j$ are already known [22] to be

$$E_N = -\frac{\kappa^2}{2(N+1)^2}, \quad N = j+1, j+2, \ldots \tag{5.15}$$

This result coincides with (5.6) under the condition that n's are non-negative even numbers, $n = 2j, 2(j+1), 2(j+2), \ldots$

6. Kinematical symmetry

We now wish to consider the angular momentum operators in the complex vector bundle $\mathcal{E}_l$. Generally speaking, the angular momentum operators are infinitesimal generators of the action of a rotation group. In our case, we have to understand how the $SO(5)$ action on $\mathbb{R}^5$ is represented in the space of cross sections in $\mathcal{E}_l$. To this end, we start with the $Sp(2)$ action on the $SU(2)$ bundle $\dot{\mathbb{R}}^8 \to \dot{\mathbb{R}}^5$ as a group of bundle automorphisms [13–15]. It is well known that $Sp(2)$, the group of 2×2 quaternionically unitary matrices, is isomorphic to $Spin(5)$, the simply connected double cover of $SO(5)$, as Lie groups.

6.1. The $Sp(2)$ action

Recall that we treat the $SU(2)$ action on $\dot{\mathbf{H}}^2$ to the left. We are to treat the $Sp(2)$ action on $\dot{\mathbf{H}}^2$ to the right, so that it may commute with the $SU(2)$ action. Let

$$\begin{aligned} u_1 &= x_0 + \mathrm{i}x_1, \quad & u_2 &= x_2 + \mathrm{i}x_3, \\ v_1 &= y_0 + \mathrm{i}y_1, \quad & v_2 &= y_2 + \mathrm{i}y_3. \end{aligned} \tag{6.1}$$

Then, from (2.1), $(X, Y) \in \dot{\mathbf{H}}^2$ is thought of as a pair of quaternions of the form

$$(u_1 - \overline{u_2}\mathrm{j}, v_1 - \overline{v_2}\mathrm{j}) = (u_1, v_1) - (\overline{u_2}, \overline{v_2})\mathrm{j}, \tag{6.2}$$

where j is a quaternion unit with the property that $\mathrm{j}^2 = -1$ and $\zeta\mathrm{j} = \mathrm{j}\overline{\zeta}$ for $\zeta \in \mathbb{C}$. Let

$$\zeta_1 = (u_1, v_1), \quad \zeta_2 = (u_2, v_2) \tag{6.3}$$

be row vectors in $\mathbb{C}^2$. Then, from (6.2), $(X, Y) \in \mathbf{H}^2$ is represented as a quaternionic row vector $\zeta_1 - \overline{\zeta}_2\mathrm{j}$ in $\mathbf{H}^2$.

Putting a quaternionic 2×2 matrix in the form $A + B\mathrm{j}$ with A and B 2×2 complex matrices, we get the right action of $A + B\mathrm{j}$ on $\mathbf{H}^2$, which can be expressed as

$$(\zeta_1, -\overline{\zeta}_2)\begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}. \tag{6.4}$$

The quaternionic unitarity condition, $(A + B\mathrm{j})(A + B\mathrm{j})^* = I_2$, implies that

$$AA^* + BB^* = I_2, \quad BA^{\mathrm{T}} - AB^{\mathrm{T}} = 0, \tag{6.5}$$

where I_2 is the 2×2 identity matrix, and the superscript asterisk * and T indicate the Hermitian conjugate and the transpose, respectively. Thus $Sp(2)$ is represented in the complex matrix form in (6.4) under the conditions (6.5). Accordingly, the action of the Lie algebra $sp(2)$ of $Sp(2)$ is put also in the form

$$(\zeta_1, -\bar{\zeta}_2)\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad \alpha + \alpha^* = 0, \ \beta = \beta^{\mathrm{T}}. \tag{6.6}$$

From this, a system of linearly independent infinitesimal generators L_{ij}, $i, j = 0, \ldots, 4$, of $Sp(2)$ are obtained, which are listed in Appendix A. We give here the commutation relations only,

$$[L_{ij}, L_{jk}] = 2L_{ik} \quad (i \neq k), \qquad [L_{ij}, L_{kl}] = 0 \quad (\neq i, j, k, l). \tag{6.7}$$

6.2. *The angular momentum operators*

Since the action of $Sp(2)$ commutes with that of $SU(2)$, it projects to an action on $\dot{\mathbb{R}}^5$ through the projection $\pi : \dot{\mathbb{R}}^8 \to \dot{\mathbb{R}}^5$. However, since (X, Y) and $(-X, -Y)$ in $\dot{\mathbf{H}}^2 \cong \dot{\mathbb{R}}^8$ project to the same point in $\dot{\mathbb{R}}^5$, the action induced on $\dot{\mathbb{R}}^5$ is that of $SO(5) \cong Spin(5)/\mathbf{Z}_2$. Therefore, to deal with the angular momentum operators as infinitesimal generators of $SO(5)$ acting on $\dot{\mathbb{R}}^5$, we are to treat the $Sp(2)$ action on $\dot{\mathbb{R}}^8$ and then to proceed to the induced action on the vector bundle $\mathcal{E}_l$. Because of the commutativity of $Sp(2)$ and $SU(2)$ actions on $\dot{\mathbb{R}}^8$, the D^l-equivariance of a V_l-valued function on $\dot{\mathbb{R}}^8$ is preserved under the $Sp(2)$ action. Hence, the $Sp(2)$ action is represented in the space of cross sections in $\mathcal{E}_l$, so that the angular momentum operators are defined to be infinitesimal generators of this $Sp(2)$ action on $\mathcal{E}_l$.

Let the infinitesimal generators L_{jk} on $\dot{\mathbb{R}}^8$ be broken up into

$$L_{jk} = 2\left(w_j\left(\frac{\partial}{\partial w_k}\right)^* - w_k\left(\frac{\partial}{\partial w_j}\right)^*\right) + 2r^2\left[\left(\frac{\partial}{\partial w_j}\right)^*, \left(\frac{\partial}{\partial w_k}\right)^*\right], \tag{6.8}$$

which can be verified by straightforward calculation along with (2.16). The angular momentum operators then turns out to be given by

$$\Lambda_{jk} := q_l^{\#-1}\frac{1}{2}L_{jk}q_l^{\#} = w_j\nabla_k - w_k\nabla_j + r^2R\left(\frac{\partial}{\partial w_j}, \frac{\partial}{\partial w_k}\right), \tag{6.9}$$

where use has been made of (3.4) and (3.6). These are global expressions of the angular momentum operators that Yang [7] gave in terms of local coordinates.

In conclusion, we define a second-order Casimir operator to be

$$L^2 = \sum_{j<k} L_{jk}^2. \tag{6.10}$$

This operator and the standard Laplacian, Δ_7, on the unit sphere S^7 are related in the following manner. Let Δ_7 be put in terms of the Cartesian coordinates:

$$\begin{aligned} \Delta_7 = (|x|^2 + |y|^2)&\sum_{\nu=0}^{3}\left(\left(\frac{\partial}{\partial x_\nu}\right)^2 + \left(\frac{\partial}{\partial y_\nu}\right)^2\right) \\ &- \left[\sum_{\nu=0}^{3}\left(x_\nu\frac{\partial}{\partial x_\nu} + y_\nu\frac{\partial}{\partial y_\nu}\right)\right]^2 - 6\sum_{\nu=0}^{3}\left(x_\nu\frac{\partial}{\partial x_\nu} + y_\nu\frac{\partial}{\partial y_\nu}\right)^2. \end{aligned} \tag{6.11}$$

Then, a straightforward calculation using (2.13), (6.10), and (6.11) results in

$$L^2 - \sum_{a=1}^{3} F_a^2 = \Delta_7, \tag{6.12}$$

which was suggested in [23].

7. Dynamical symmetry

In Theorem 5.3, we have referred to the space, $S(E_n; l)$, of D^l-equivariant functions which is a subspace of the eigenspace $S(E_n)$ for $\widehat{H}$. In this section, we show that $S(E_n; l)$ can be actually formed out of the eigenspace $S(E_n)$ and further consider what group acts on $S(E_n; l)$. In order to form $S(E_n; l)$, we have only to pick up carrier spaces for D^l out of the eigenspace $S(E_n)$ of $\widehat{H}$.

7.1. Picking up carrier spaces for D^l

In view of the close relation (5.4) between the quantized conformal Kepler problem and the harmonic oscillator, we are first to pick up carrier spaces for D^l out of the eigenspace for the harmonic oscillator. Let

$$\begin{aligned} a_\nu &= \frac{1}{\sqrt{2\lambda}}\left(\lambda x_\nu + \frac{\partial}{\partial x_\nu}\right), & a_\nu^\dagger &= \frac{1}{\sqrt{2\lambda}}\left(\lambda x_\nu - \frac{\partial}{\partial x_\nu}\right), \\ b_\nu &= \frac{1}{\sqrt{2\lambda}}\left(\lambda y_\nu + \frac{\partial}{\partial y_\nu}\right), & b_\nu^\dagger &= \frac{1}{\sqrt{2\lambda}}\left(\lambda y_\nu - \frac{\partial}{\partial y_\nu}\right), \end{aligned} \tag{7.1}$$

where $\nu = 0, \ldots, 3$. Further, set

$$\begin{aligned} A_1^\dagger &= \frac{1}{\sqrt{2}}(a_0^\dagger - \mathrm{i}a_1^\dagger), & A_5^\dagger &= \frac{1}{\sqrt{2}}(b_0^\dagger - \mathrm{i}b_1^\dagger), \\ A_2^\dagger &= \frac{1}{\sqrt{2}}(a_2^\dagger - \mathrm{i}a_3^\dagger), & A_6^\dagger &= \frac{1}{\sqrt{2}}(b_2^\dagger - \mathrm{i}b_3^\dagger), \\ A_3^\dagger &= -\frac{1}{\sqrt{2}}(a_2^\dagger + \mathrm{i}a_3^\dagger), & A_7^\dagger &= -\frac{1}{\sqrt{2}}(b_2^\dagger + \mathrm{i}b_3^\dagger), \\ A_4^\dagger &= \frac{1}{\sqrt{2}}(a_0^\dagger + \mathrm{i}a_1^\dagger), & A_8^\dagger &= \frac{1}{\sqrt{2}}(b_0^\dagger + \mathrm{i}b_1^\dagger), \end{aligned} \tag{7.2}$$

and let A_k, $k = 1, 2, \ldots, 8$, be the dual operators to $A_k^\dagger$, respectively. These operators are creation and annihilation operators satisfying the canonical commutation relations,

$$[A_k, A_j^\dagger] = \delta_{kj}, \quad k, j = 1, \ldots, 8, \tag{7.3}$$

with the others vanishing. In terms of $A_k^\dagger$, the eigenfunctions of the harmonic oscillator $\widehat{K}_\lambda$ are expressed as

$$(\mathbf{n}!)^{-1/2}(A_1^\dagger)^{n_1}(A_2^\dagger)^{n_2}(A_3^\dagger)^{n_3}(A_4^\dagger)^{n_4}(A_5^\dagger)^{n_5}(A_6^\dagger)^{n_6}(A_7^\dagger)^{n_7}(A_8^\dagger)^{n_8}|0\rangle,$$

$$\mathbf{n}! := n_1! \cdots n_8!, \tag{7.4}$$

where $|0\rangle$ is the normalized ground state. These functions form a complete basis of $L^2(\mathbb{R}^8)$.

The $SU(2)$ action on $\mathbf{H}^2 \cong \mathbb{R}^8$ given by (2.8) is represented unitarily in $L^2(\mathbb{R}^8)$:

$$(U_g f)(q) := f(\phi_{g^{-1}}(q)), \quad q \in \mathbb{R}^8, \quad g \in SU(2). \tag{7.5}$$

From the definitions (7.1) and (7.2), the U_g induces linear transformations on the operators $A_k^\dagger$ as follows:

$$U_g A^\dagger U_g^{-1} = \mathrm{diag}(g^{\mathrm{T}}, g^{\mathrm{T}}, g^{\mathrm{T}}, g^{\mathrm{T}})A^\dagger, \tag{7.6}$$

where $A^\dagger = (A_1^\dagger, A_2^\dagger, \ldots, A_8^\dagger)^{\mathrm{T}}$ is a vector of operators, and the diag$(\cdot)$ is a block diagonal 8×8 matrix. On inserting the right-hand sides of (7.6) into (7.4) and expanding the resultant operator polynomial, the action of U_g is represented with respect to the basis functions (7.4). Then, one finds that the eigenspace designated by $n = n_1 + n_2 + \cdots + n_8$ includes the carrier space for the tensor product representation

$$D^{p_1/2} \otimes D^{p_2/2} \otimes D^{p_3/2} \otimes D^{p_4/2} \tag{7.7}$$

with

$$\begin{aligned} &n_1 + n_2 = p_1, \; n_3 + n_4 = p_2, \\ &n_5 + n_6 = p_3, \; n_7 + n_8 = p_4, \qquad p_1 + p_2 + p_3 + p_4 = n. \end{aligned} \tag{7.8}$$

Decomposing the tensor product (7.7) into a Clebsch–Gordan series, we obtain a series of irreducible representations designated by the numbers of the form $l := \frac{1}{2}n - N$, where N's are non-negative integers. Thus we have

$$n = 2N + 2l. \tag{7.9}$$

On account of (7.6) and the Clebsch–Gordan series, basis functions f_m ($|m| \leq l$) in the carrier space for D^l are subject to the transformation

$$f_m(\phi_g(q)) = \sum_{|k|\leq l} f_k(q) D^l_{km}(g^{-1}), \quad q \in \mathbb{R}^8. \tag{7.10}$$

The right-hand side of (7.10) defines a left $SU(2)$ action on the carrier space, which is denoted by $(D^l(g)f_m)(q)$. Thus the carrier space for D^l turns out to be a space of D^l-equivariant functions. Hence we have the following.

Lemma 7.1. *Possible carrier spaces for the representation of $SU(2)$ in the eigenspace of the harmonic oscillator assigned by n are spaces of D^l-equivariant functions, where l is an integer or a half-integer, according as n is even or odd, together with $n \geq 2l$. We have to note also that each eigenspace designated by n is decomposed into a direct sum of spaces of D^l-equivariant functions with suitable l's, $0 \leq l \leq \frac{1}{2}n$.*

7.2. SU(4) symmetry

According to Iwai [24], $SU(8)$ acts on $L^2(\mathbb{R}^8)$ unitarily as follows: let W_G be a unitary operator corresponding to a matrix G in $SU(8)$. The action of W_G on a base function f in (7.4) is then given by

$$W_G f(q) = (\mathbf{n}!)^{-1/2}(G^{\mathrm{T}}A^\dagger)_1^{n_1}(G^{\mathrm{T}}A^\dagger)_2^{n_2}\cdots(G^{\mathrm{T}}A^\dagger)_8^{n_8}|0\rangle, \tag{7.11}$$

where $(G^{\mathrm{T}}A^\dagger)_j$, $j = 1, \ldots, 8$, denotes the jth component of the vector, $G^{\mathrm{T}}A^\dagger$, of operators. Note that the W_G induces the transformation of $A^\dagger$,

$$W_G A^\dagger W_G^{-1} = G^{\mathrm{T}}A^\dagger. \tag{7.12}$$

We mention here that Eq. (7.12) covers Eq. (7.6). In fact, for $G = \mathrm{diag}(g, g, g, g)$ with $g \in SU(2)$, W_G becomes U_g.

If we set $\lambda = 4\kappa/(n+4)$ in (7.1), the functions (7.4) become eigenfunctions of $\widehat{H}$ with eigenvalue E_n on account of Lemma 5.1, and therefore W_G gives rise to an action of $SU(8)$ on the eigenspace $S(E_n)$ for the quantized conformal Kepler problem of eigenvalue E_n. We denote this action by $W_G^{(n)} := W_G|_{S(E_n)}$, which can be shown to be a unitary operator on $S(E_n)$ with respect to the inner product (5.2). The proof can be made through the same calculation as that in [6]. In an analogous manner, we can define $U_g^{(n)} := U_g|_{S(E_n)}$ acting on $S(E_n)$. Further, the eigenspace $S(E_n)$ is decomposed into a direct sum of spaces $S(E_n; l)$ of D^l-equivariant functions.

Lemma 7.2. *For all possible l, the space $S(E_n; l)$ of D^l-equivariant eigenfunctions is an invariant subspace for $W_G^{(n)}$ if and only if $W_G^{(n)}$ and $U_g^{(n)}$ commute for any $g \in SU(2)$.*

Proof. We notice first that from (7.10) the D^l-equivariance is expressed also as $U_g^{(n)} f_m = D^l(g^{-1}) f_m$ for basis functions $f_m \in S(E_n; l)$. Suppose $W_G^{(n)}$ and $U_g^{(n)}$ commute. Then, we obtain

$$\begin{aligned}(W_G^{(n)} f_m)(\phi_g(q)) &= U_{g^{-1}}^{(n)}(W_G^{(n)} f_m)(q) = W_G^{(n)} U_{g^{-1}}^{(n)} f_m(q) = W_G^{(n)} D^l(g) f_m(q)\\ &= W_G^{(n)} \sum_{|k|\le l} f_k(q) D^l_{km}(g^{-1}) = \sum_{|k|\le l}(W_G^{(n)} f_k)(q) D^l_{km}(g^{-1})\\ &= D^l(g) W_G^{(n)} f_m(q).\end{aligned} \tag{7.13}$$

This verifies that $W_G^{(n)} f_m$ is D^l-equivariant, so that $S(E_n; l)$ is an invariant subspace for $W_G^{(n)}$. Conversely, if $S(E_n; l)$ is invariant under $W_G^{(n)}$ for all possible l, then $W_G^{(n)}$ and $U_g^{(n)}$ prove to commute on $S(E_n)$. This ends the proof. □

We are to study what group should act on $S(E_n; l)$. To this end, we are looking for a subgroup of $SU(8)$, which consists of $SU(8)$ matrices commuting with all the 8×8 matrices $\mathrm{diag}(g, g, g, g)$, $g \in SU(2)$. A straightforward calculation shows that any 8×8 matrix commutative with $\mathrm{diag}(g, g, g, g)$ must take the tensor product form $C \otimes I_2$, where $C =$

$(c_{\mu\nu})_{\mu,\nu=1,\dots,4}$, and I_2 being the 2×2 identity matrix. Since the matrix $C \otimes I_2$ is required further to be in $SU(8)$, and since $\det(C \otimes I_2) = (\det C)^2$, C is shown to be subject to

$$\sum_{\lambda=1}^{4} c_{\mu\lambda}\bar{c}_{\nu\lambda} = \delta_{\mu\nu}, \quad \det(c_{\mu\nu}) = \pm 1. \tag{7.14}$$

The identity component of the group determined by (7.14) is, of course, $SU(4)$, which we choose to take. Thus the whole group $SU(8)$ reduces to the subgroup $SU(4)$, the action of which is represented as unitary operators $W_C^{(n)}$ on $S(E_n; l)$, where we have used C instead of $C \otimes I_2$ for the sake of notational simplicity. Clearly, these $W_C^{(n)}$ commute with $U_g^{(n)}$, $g \in SU(2)$, so that the $SU(4)$ acts on $S(E_n; l)$ for all possible l. Since $S(E_n; l)$ is in one-to-one correspondence with the eigenspace of the quantized $SU(2)$ Kepler problem, we have the following theorem.

Theorem 7.3. *Each of the eigenspaces $q_l^{\#-1} S(E_n; l)$ of the quantized $SU(2)$ Kepler problem admits a unitary representation of the symmetry group $SU(4)$.*

In classical mechanics, we have already shown that the $SU(2)$ Kepler problem admits the symmetry group $SU(4)$, which acts on each energy manifold of negative energy [10]. Theorem 7.3 is then a quantum version of the classical symmetry.

In Appendix B, we give a basis of the Lie algebra $su(4)$, J_{jk} and Q_n with $j, k, n = 0, 1, \dots, 4$, $j < k$, represented in the 8×8 matrix form. The commutation relations among them are calculated as

$$\begin{aligned}
[J_{ij}, J_{jk}] &= -J_{ik} \quad (i \neq j), \\
[J_{ij}, J_{kl}] &= 0 \quad (\neq i, j, k, l), \\
[Q_j, Q_k] &= J_{jk}, \\
[J_{jk}, Q_n] &= \delta_{jn} Q_k - \delta_{kn} Q_j \quad (j \neq k).
\end{aligned} \tag{7.15}$$

We notice here that $SU(4) \cong Spin(6)$, which is shown in [25], for example, so that (7.15) is looked upon as the commutation relations of $so(6)$. Note also that the J_{jk}'s form a basis of the Lie algebra $sp(2) \cong so(5)$ of $Sp(2)$.

8. Generators of the symmetry group

We have obtained in Section 7 the operators $W_C^{(n)}$ for the symmetry group $SU(4)$, which act on the eigenspace $S(E_n)$ for the quantized conformal Kepler problem of negative energy. We denote by $\widehat{F}^{(n)}$ one of infinitesimal generators of $W_C^{(n)}$ on $S(E_n)$. Since $W_C^{(n)}$ and $U_g^{(n)}$ commute, one has $U_g^{(n)} \widehat{F}^{(n)} U_g^{(n)-1} = \widehat{F}^{(n)}$ for the infinitesimal generator $\widehat{F}^{(n)}$. If $\widehat{F}^{(n)}$'s are put together (in n) to form an operator $\widehat{F}$ (densely defined in the whole space $L^2(\mathbb{R}^8; 4r\,\mathrm{d}x\,\mathrm{d}y)$), $\widehat{F}$ will then satisfy

$$U_g \widehat{F} U_g^{-1} = \widehat{F}, \quad g \in SU(2). \tag{8.1}$$

For operators $\widehat{F}$ satisfying (8.1), we can define reduced operators $\widehat{F}_l$ acting on Γ_l through

$$\widehat{F}_l = q_l^{\#-1}\widehat{F}q_l^{\#}, \tag{8.2}$$

which will form infinitesimal generators of the symmetry group $SU(4)$ for the quantized $SU(2)$ Kepler problem of negative energy.

Our task is then to find infinitesimal generators of $W_C^{(n)}$ for $C \in SU(4)$. Incidentally, the previous article [24] of ours shows that such generators are given by

$$\widehat{F}^{(n)} = \frac{\lambda}{2i}\sum_{j,k=1}^{8} F_{kj}A_k A_j^{\dagger}, \tag{8.3}$$

where $F = (F_{ij}) \in su(4)$ takes the matrix form given in Appendix B. If we take (F_{ij}) to be a base matrix Q_1 given in Appendix B, the operator (8.3) is written out to give

$$\widehat{F}^{(n)} = \frac{1}{4}\Big(- \frac{\partial^2}{\partial x_0 \partial y_1} + \frac{\partial^2}{\partial x_1 \partial y_0} - \frac{\partial^2}{\partial x_2 \partial y_3} + \frac{\partial^2}{\partial x_3 \partial y_2}$$
$$+(x_0y_1 - x_1y_0 + x_2y_3 - x_3y_2)\lambda^2\Big),$$

where $\lambda = 4\kappa/(n+4)$. To form an operator $\widehat{F}$ from $F^{(n)}$'s, we replace the number λ^2 by $-8\widehat{H}$ according to the relation $\widehat{H} = -\frac{1}{8}\lambda^2$ which holds on the eigenspace $S(E_n)$. As another example, we take $F = J_{01}$ (see Appendix B). Then Eq. (8.3) results in the operator

$$\frac{\lambda}{4i}\left(x_1\frac{\partial}{\partial x_0} - x_0\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_3} - y_1\frac{\partial}{\partial y_0} + y_0\frac{\partial}{\partial y_1} - y_3\frac{\partial}{\partial y_2} + y_2\frac{\partial}{\partial y_3}\right).$$

Factoring out $\frac{1}{2}\lambda$ in this operator, we find an angular momentum operator. Applying the same method to the other basis matrices of $su(4)$ provides the infinitesimal generators (densely defined on the whole space $L^2(\dot{\mathbb{R}}^8; 4r\,\mathrm{d}x\,\mathrm{d}y)$) as follows:

$$\widehat{L}_{jk} = \frac{1}{2i}L_{jk} \quad (j,k = 0,\ldots,4),$$

$$\widehat{D}_0 = -\frac{1}{4}\left(\frac{\partial^2}{\partial x_0 \partial y_0} + \frac{\partial^2}{\partial x_1 \partial y_1} + \frac{\partial^2}{\partial x_2 \partial y_2} + \frac{\partial^2}{\partial x_3 \partial y_3}\right)$$
$$-2(x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3)\widehat{H},$$

$$\widehat{D}_1 = -\frac{1}{4}\left(\frac{\partial^2}{\partial x_0 \partial y_1} - \frac{\partial^2}{\partial x_1 \partial y_0} + \frac{\partial^2}{\partial x_2 \partial y_3} - \frac{\partial^2}{\partial x_3 \partial y_2}\right)$$
$$-2(x_0y_1 - x_1y_0 + x_2y_3 - x_3y_2)\widehat{H},$$

$$\widehat{D}_2 = -\frac{1}{4}\left(\frac{\partial^2}{\partial x_0 \partial y_2} - \frac{\partial^2}{\partial x_2 \partial y_0} + \frac{\partial^2}{\partial x_3 \partial y_1} - \frac{\partial^2}{\partial x_1 \partial y_3}\right) \tag{8.4}$$

$$-2(x_0 y_2 - x_2 y_0 + x_3 y_1 - x_1 y_3)\widehat{H},$$

$$\widehat{D}_3 = -\frac{1}{4}\left(\frac{\partial^2}{\partial x_0 \partial y_3} - \frac{\partial^2}{\partial x_3 \partial y_0} + \frac{\partial^2}{\partial x_1 \partial y_2} - \frac{\partial^2}{\partial x_2 \partial y_1}\right)$$

$$-2(x_0 y_3 - x_3 y_0 + x_1 y_2 - x_2 y_1)\widehat{H},$$

$$\widehat{D}_4 = -\frac{1}{8}\left(\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial y_0^2} - \frac{\partial^2}{\partial y_1^2} - \frac{\partial^2}{\partial y_2^2} - \frac{\partial^2}{\partial y_3^2}\right)$$

$$-(x_0^2 + x_1^2 + x_2^2 + x_3^2 - y_0^2 - y_1^2 - y_2^2 - y_3^2)\widehat{H},$$

where L_{jk} are the infinitesimal generators listed in Appendix A. These are symmetric operators on $C_0^\infty(\mathbb{R}^8)$ with respect to the inner product (5.2). It is straightforward to show that the operators (8.4) satisfy (8.1).

The commutation relations among them are shown to be given by

$$\begin{aligned}
&[\widehat{L}_{ij}, \widehat{L}_{jk}] = -i\widehat{L}_{ik} && (i \neq k),\\
&[\widehat{L}_{ij}, \widehat{L}_{kl}] = 0 && (\neq i, j, k, l),\\
&[\widehat{D}_j, \widehat{D}_k] = i\widehat{L}_{jk}(-2\widehat{H}) && (j \neq k),\\
&[\widehat{L}_{jk}, \widehat{D}_n] = i(\delta_{jn}\widehat{D}_k - \delta_{kn}\widehat{D}_j) && (j \neq k),
\end{aligned} \tag{8.5}$$

where i, j, k and n range over $0, \ldots, 4$. Now that we have found symmetry operators satisfying (8.1), we are to reduce these operators to obtain symmetry operators for the quantized $SU(2)$ Kepler problem. By using (2.10) and (6.8), we put (8.4) in the form

$$\begin{aligned}
\widehat{L}_{jk} &= \frac{1}{i}\left(w_j\left(\frac{\partial}{\partial w_k}\right)^* - w_k\left(\frac{\partial}{\partial w_j}\right)^* + r^2\left[\left(\frac{\partial}{\partial w_j}\right)^*, \left(\frac{\partial}{\partial w_k}\right)^*\right]\right),\\
\widehat{D}_n &= \frac{1}{2i}\sum_{k\neq n}\left(\widehat{L}_{kn}\left(\frac{\partial}{\partial w_k}\right)^* + \left(\frac{\partial}{\partial w_k}\right)^*\widehat{L}_{kn}\right) + \frac{\kappa}{r}w_n,
\end{aligned} \tag{8.6}$$

which can be verified by straightforward but lengthy calculation. Applying (8.2)–(8.6) together with (3.4) and (3.6), we obtain the reduced operators

$$\begin{aligned}
[\widehat{L}_{jk}]_l &= \frac{1}{i}\left(w_j\nabla_k - w_k\nabla_j + r^2 R\left(\frac{\partial}{\partial w_j}, \frac{\partial}{\partial w_k}\right)\right),\\
[\widehat{D}_n]_l &= \frac{1}{2i}\sum_{k\neq n}\left([\widehat{L}_{kn}]_l\nabla_k + \nabla_k[\widehat{L}_{kn}]_l\right) + \frac{\kappa}{r}w_n,
\end{aligned} \tag{8.7}$$

where $[\cdot]_l$ indicates the operator acting on cross sections in $\mathcal{E}_l$. The commutation relations that the symmetry operators (8.7) obey are the same as (8.5). We note further that the symmetry operators (8.7) apply for the quantized $SU(2)$ Kepler problem of whole energies. Thus we have the following.

Theorem 8.1. *The $SU(2)$-invariant operators (8.4) for the quantized conformal Kepler problem are reduced to the symmetry operators (8.7) for the quantized $SU(2)$ Kepler problem $(\Gamma_l, \widehat{H}_l)$. The $[\widehat{L}_{jk}]_l$ are the angular momentum operators and the $[\widehat{D}_n]_l$ the Runge–Lenz-like operators. These operators form the Lie algebra $so(6)$, $e(5)$, or $so(1,5)$, according to whether the energy $\widehat{H}_l$ is negative, zero, or positive, where $e(5)$ is the Lie algebra of the group $E(5)$ of Euclidean motions in $\mathbb{R}^5$.*

Appendix A

Linearly independent generators for $Sp(2)$:

	$\partial/\partial x_0$	$\partial/\partial x_1$	$\partial/\partial x_2$	$\partial/\partial x_3$	$\partial/\partial y_0$	$\partial/\partial y_1$	$\partial/\partial y_2$	$\partial/\partial y_3$
L_{01}	x_1	$-x_0$	x_3	$-x_2$	$-y_1$	y_0	$-y_3$	y_2
L_{02}	x_2	$-x_3$	$-x_0$	x_1	$-y_2$	y_3	y_0	$-y_1$
L_{03}	x_3	x_2	$-x_1$	$-x_0$	$-y_3$	$-y_2$	y_1	y_0
L_{12}	$-x_3$	$-x_2$	x_1	x_0	$-y_3$	$-y_2$	y_1	y_0
L_{23}	$-x_1$	x_0	$-x_3$	x_2	$-y_1$	y_0	$-y_3$	y_2
L_{31}	$-x_2$	x_3	x_0	$-x_1$	$-y_2$	y_3	y_0	$-y_1$
L_{04}	y_0	y_1	y_2	y_3	$-x_0$	$-x_1$	$-x_2$	$-x_3$
L_{14}	y_1	$-y_0$	y_3	$-y_2$	x_1	$-x_0$	x_3	$-x_2$
L_{24}	y_2	$-y_3$	$-y_0$	y_1	x_2	$-x_3$	$-x_0$	x_1
L_{34}	y_3	y_2	$-y_1$	$-y_0$	x_3	x_2	$-x_1$	$-x_0$

Appendix B

The bases of $su(4)$:

$$\sum_{n=0}^{4} \beta_n Q_n + \sum_{j<k} \alpha_{jk} J_{jk}$$
$$= \frac{1}{2} \begin{pmatrix} \mathrm{i}\beta_4 I_2 & 0 & (\mathrm{i}\beta_0 - \beta_1) I_2 & (-\mathrm{i}\beta_2 - \beta_3) I_2 \\ 0 & \mathrm{i}\beta_4 I_2 & (\mathrm{i}\beta_2 - \beta_3) I_2 & (\mathrm{i}\beta_0 + \beta_1) I_2 \\ (\mathrm{i}\beta_0 + \beta_1) I_2 & (\mathrm{i}\beta_2 + \beta_3) I_2 & -\mathrm{i}\beta_4 I_2 & 0 \\ (-\mathrm{i}\beta_2 + \beta_3) I_2 & (\mathrm{i}\beta_0 - \beta_1) I_2 & 0 & -\mathrm{i}\beta_4 I_2 \end{pmatrix}$$

$$
+\frac{1}{2}\begin{pmatrix} 0 & (-\alpha_{02}+\alpha_{31})I_2 & \alpha_{04}I_2 & -\alpha_{24}I_2 \\ (\alpha_{02}-\alpha_{31})I_2 & 0 & \alpha_{24}I_2 & \alpha_{04}I_2 \\ -\alpha_{04}I_2 & -\alpha_{24}I_2 & 0 & (\alpha_{02}+\alpha_{31})I_2 \\ \alpha_{24}I_2 & -\alpha_{04}I_2 & -(\alpha_{02}+\alpha_{31})I_2 & 0 \end{pmatrix}
$$

$$
+\frac{\mathrm{i}}{2}\begin{pmatrix} (\alpha_{01}-\alpha_{23})I_2 & (\alpha_{03}-\alpha_{12})I_2 & \alpha_{14}I_2 & \alpha_{34}I_2 \\ (\alpha_{03}-\alpha_{12})I_2 & (-\alpha_{01}+\alpha_{23})I_2 & \alpha_{34}I_2 & -\alpha_{14}I_2 \\ \alpha_{14}I_2 & \alpha_{34}I_2 & -(\alpha_{01}+\alpha_{23})I_2 & -(\alpha_{03}+\alpha_{12})I_2 \\ \alpha_{34}I_2 & -\alpha_{14}I_2 & -(\alpha_{03}+\alpha_{12})I_2 & (\alpha_{01}+\alpha_{23})I_2 \end{pmatrix}
$$

where I_2 is the 2×2 identity matrix.

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